Solution to Math4230 Tutorial 9

1. Consider the problem

$$\min f_1(x) - f_2(Qx)$$
$$s.t.\ x \in \mathbb{R}^n,$$

where $Q \in \mathbb{R}^{m\times n}$ and $f_1 : \mathbb{R}^n \mapsto (-\infty, \infty]$ and $f_2 : \mathbb{R}^m \mapsto [-\infty, \infty)$ are extended real-valued functions. Show that the corresponding max crossing problem is

$$\max h_2(\mu) - h_1(Q^T\mu)$$
$$s.t.\ \mu \in \mathbb{R}^n,$$

where $h_1(Q^T\mu) = \sup_{y\in\mathbb{R}^n} \left\{y^TQ^T\mu - f_1(y)\right\}$, $h_2(\mu) = \inf_{z\in\mathbb{R}^m} \left\{z^T\mu - f_2(z)\right\}$.

Hint: Consider $F(x, u) = f_1(x) - f_2(Qx + u)$.

Solution:
Here $p(\mu) = \inf_{y\in\mathbb{R}^n} F(y, \mu)$ and

$$\begin{aligned}
q(\mu) &= \inf_{z\in\mathbb{R}^m} \{p(z) + \mu^Tz\} \\
&= \inf_{(y,z)\in(\mathbb{R}^n,\mathbb{R}^m)} \{f_1(y) - f_2(Qy + z) + \mu^Tz\} \\
&= \inf_{(y,z)\in(\mathbb{R}^n,\mathbb{R}^m)} \{f_1(y) - f_2(z) + \mu^Tz - \mu^TQy\} \\
&= \inf_{y\in\mathbb{R}^n} \{f_1(y) - \mu^TQy\} + \inf_{z\in\mathbb{R}^m} \{f_2(z) + \mu^Tz\} \\
&= h_2(\mu) - h_1(Q^T\mu)
\end{aligned}$$

2. Consider the problem

$$\min f(x)$$
$$s.t.\ x \in X, \quad e_i^Tx = d_i, \quad i = 1, \cdots, m,$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function, X is a convex set, and e_i and d_i are given vectors and scalars, respectively. Consider the min common/max crossing framework where M is the subset of $\mathbb{R}^{m+1}$ given by

$$M = \left\{(e_1^Tx - d_1, \cdots, e_m^Tx - d_m, f(x)) \mid x \in X\right\}.$$

(a) Derive the corresponding max crossing problem;
(b) Show that the corresponding set $\bar{M}$ is convex.

Solution

(a) The corresponding max crossing problem is given by

$$q^* = \sup_{\mu \in \Re^m} q(\mu),$$

where $q(\mu)$ is given by

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\} = \inf_{x\in X} \left\{ f(x) + \sum_{i=1}^{m} \mu_i (e_i' x - d_i) \right\}.$$

(b) Consider the set

$$\overline{M} = \left\{ \big(u_1, \ldots, u_m, w\big) \mid \exists\, x \in X \text{ such that } e_i' x - d_i = u_i,\ \forall\, i,\ f(x) \leq w \right\}.$$

We show that $\overline{M}$ is convex. To this end, we consider vectors $(u, w) \in \overline{M}$ and $(\tilde{u}, \tilde{w}) \in \overline{M}$, and we show that their convex combinations lie in $\overline{M}$. The definition of $\overline{M}$ implies that for some $x \in X$ and $\tilde{x} \in X$, we have

$$f(x) \leq w, \qquad e_i' x - d_i = u_i, \quad i = 1, \ldots, m,$$
$$f(\tilde{x}) \leq \tilde{w}, \qquad e_i' \tilde{x} - d_i = \tilde{u}_i, \quad i = 1, \ldots, m.$$

For any $\alpha \in [0, 1]$, we multiply these relations with α and 1-α, respectively, and add. By using the convexity of f, we obtain

$$f\big(\alpha x + (1-\alpha)\tilde{x}\big) \leq \alpha f(x) + (1-\alpha) f(\tilde{x}) \leq \alpha w + (1-\alpha)\tilde{w},$$

$$e_i'\big(\alpha x + (1-\alpha)\tilde{x}\big) - d_i = \alpha u_i + (1-\alpha)\tilde{u}_i, \quad i = 1, \ldots, m.$$

In view of the convexity of X, we have $\alpha x + (1-\alpha)\tilde{x} \in X$, so these equations imply that the convex combination of (u, w) and $(\tilde{u}, \tilde{w})$ belongs to $\overline{M}$, thus proving that $\overline{M}$ is convex.

3. Let $f : X \mapsto [-\infty, \infty]$ be a function. Prove that:

$$\inf_{x\in X} f(x) = \inf_{x\in X} (\text{cl } f)(x) = \inf_{x\in \mathbb{R}^n} F(x),$$

where $F(x) = \inf\{w | (x, w) \in \text{conv}(\text{epi}(f))\}$. Furthermore, any vector that attains the infimum of f over X also attains the infimum of cl$f$ and F.

Solution Please refer to proof of Proposition 1.3.13 in Appendix.

Appendix

Proof of Proposition 1.3.13:

Proof: If $\text{epi}(f)$ is empty, i.e., $f(x) = \infty$ for all x, the results trivially hold. Assume that $\text{epi}(f)$ is nonempty, and let $f^* = \inf_{x \in \Re^n} (\text{cl}\, f)(x)$. For any sequence $\{(\overline{x}_k, \overline{w}_k)\} \subset \text{cl}\big(\text{epi}(f)\big)$ with $\overline{w}_k \to f^*$, we can construct a sequence $\{(x_k, w_k)\} \subset \text{epi}(f)$ such that $|w_k - \overline{w}_k| \to 0$, so that $w_k \to f^*$. Since $x_k \in X$, $f(x_k) \leq w_k$, we have

$$\limsup_{k \to \infty} f(x_k) \leq f^* \leq (\text{cl}\, f)(x) \leq f(x), \qquad \forall\, x \in X,$$

so that

$$\inf_{x \in X} f(x) = \inf_{x \in X} (\text{cl}\, f)(x) = \inf_{x \in \Re^n} (\text{cl}\, f)(x).$$

Choose $\{(x_k, w_k)\} \subset \text{conv}\big(\text{epi}(f)\big)$ with $w_k \to \inf_{x \in \Re^n} F(x)$. Each (x_k, w_k) is a convex combination of vectors from $\text{epi}(f)$, so that $w_k \geq \inf_{x \in X} f(x)$. Hence $\inf_{x \in \Re^n} F(x) \geq \inf_{x \in X} f(x)$. On the other hand, we have $F(x) \leq f(x)$ for all $x \in X$, so it follows that $\inf_{x \in \Re^n} F(x) = \inf_{x \in X} f(x)$. Since $\check{\text{cl}}\, f$ is the closure of F, it also follows (based on what was shown in the preceding paragraph) that $\inf_{x \in \Re^n} (\check{\text{cl}}\, f)(x) = \inf_{x \in \Re^n} F(x)$.

We have $f(x) \geq (\text{cl}\, f)(x)$ for all x, so if x^* attains the infimum of f,

$$\inf_{x \in \Re^n} (\text{cl}\, f)(x) = \inf_{x \in X} f(x) = f(x^*) \geq (\text{cl}\, f)(x^*),$$

showing that x^* attains the infimum of $\text{cl}\, f$. Similarly, x^* attains the infimum of F and $\check{\text{cl}}\, f$. **Q.E.D.**